Bifurcation of elliptical equilibria

R. A. Clemente and J. L. Milovich

Comisión Nacional de Energía Atómica, Depto. Prospectiva y Estudios Especiales, División Fusión Nuclear, 1429 Buenos Aires, Argentina

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It is shown that a bifurcation of the equilibrium solutions for straight elliptical plasma columns with a diffuse current profile exists for a suitable choice of the currents in the external conductors.

Theoretical and experimental studies[1-3] have shown the possibility of improving on β and stability in tokamaks by elongating the plasma cross section. In the large aspect ratio approximation, the equilibrium problem of elongated configurations reduces to the analysis of an elliptical plasma column in an external magnetic field produced by four conductors located symmetrically around the plasma, at a distance d very large compared with the characteristic transverse dimension of the plasma ($a^2 + b^2 \ll d^2$, see Fig. 1), carrying currents I_1 and $-I_2$ (Refs. 4–6).

In the case where $I_1 = I_2$, for a constant longitudinal current density model, Strauss[4] has found that the elliptical equilibrium is not uniquely determined by the boundary conditions, giving rise to two classes of configurations with different semi-axes ratios (b/a), for given values of the parameters (external currents, excluded poloidal flux, etc.). One class is characterized by $b/a \leqslant 2.9$ and the other by $b/a \geqslant 2.9$. The elongation which separates the two classes corresponds to a bifurcation point in parameter space, and it has been shown by Thyagaraja and Haas[7] to correspond to a threshold for an $m = 2$ (poloidal number) secular instability for the Strauss model.

Thomas and Haas,[6] using numerical techniques, have looked for a bifurcation point in the case of a peaked current profile, without finding it, since the separatrix enters the plasma before reaching bifurcation at the elongation of $b/a \simeq 2.1$.

In the present case we have considered the same model of Thomas and Haas analytically, allowing I_1 to be different from I_2. Then, the z component of the potential vector of the conductors $A_z = \psi_w$ near the origin of the coordinate system may be written as

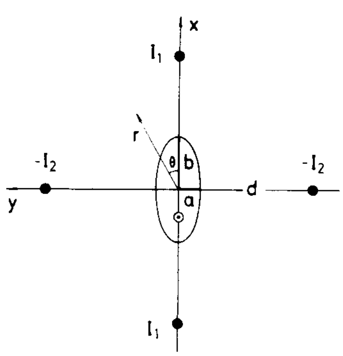

FIG. 1. Elliptic equilibrium and coordinate system.

$$\psi_w = \frac{2(I_1 + I_2)}{cd^2} r^2 \cos 2\theta + \frac{(I_1 - I_2)}{cd^4} r^4 \cos 4\theta + O\left(\frac{r^6}{d^6}\right). \tag{1}$$

Inside the plasma $A_z = \psi_{pl}$ is governed by the equation

$$\nabla^2 \psi_{pl} = -\frac{4\pi}{c} j_z , \tag{2}$$

and outside the plasma, $A_z = \psi_v$ must satisfy the equation

$$\nabla^2 \psi_v = 0 . \tag{3}$$

In order to solve Eq. (2) we make the hypothesis that the current density is such that $4\pi j_z/c = k^2 \psi_{pl}$, which corresponds to a diffuse current profile. Our boundary conditions at the plasma-vacuum interface will be $\psi_{pl} = \psi_v = 0$ and $\nabla\psi_{pl} = \nabla\psi_v$. Then, in elliptic coordinates μ, θ [$x = (\alpha \cosh \mu \cos \theta)/2$; $y = (\alpha \sinh \mu \sin \theta)/2$], the general solutions of (2) and (3), which are symmetric with respect to the axis of the ellipse, are

$$\psi_{pl} = \sum_{n=0}^{\infty} C_{2n} S_{e_{2n}}(q, \theta) J_{e_{2n}}(q, \mu) , \tag{4}$$

$$\psi_v = A_0 + A_1 \mu + \sum_{n=1}^{\infty} [A_{2n} \sinh 2n(\mu - \mu_0) + E_{2n} \cosh 2n(\mu - \mu_0)] \cos 2n\theta , \tag{5}$$

where $q = \alpha^2 k^2/16$, μ_0, D_i, A_i, E_i are constants and $S_{e_{2n}}$, $J_{e_{2n}}$ are the even angular and radial Mathieu functions of order $2n$, respectively. If we choose, as the boundary contour, an ellipse $\mu = \mu_0$, then in solution (4) only one harmonic should be considered and in solution (5), $E_{2i} = 0$. Avoiding zeros in the angular part of ψ_{pl}, the solutions are

$$\psi_{pl} = \frac{D_0 S_{e0}(q, \theta) J_{e0}(q, \mu)}{S_{e0}(q, \pi/2) J_{e0}(q, 0)} , \tag{6}$$

$$\psi_v = A_1(\mu - \mu_0) + \sum_{n=1}^{\infty} A_{2n} \sinh 2n(\mu - \mu_0) \cos 2n\theta , \tag{7}$$

with μ_0 corresponding to the first zero of $J_{e0}(q, \mu)$. Here, we have changed the normalizing factor in such a way that ψ_{pl} $(x = y = 0) = D_0$. Equating the normal derivatives of the two solutions at $\mu = \mu_0$, and using the Fourier expansion for

$$S_{e0}(q, \theta) = \sum_{n=0}^{\infty} B_{2n}^{(0)}(q) \cos 2n\theta ,$$

the following relations can be obtained

$$\frac{D_0 B_0^{(0)}(q) J'_{e0}(q, \mu_0)}{S_{e0}(q, \pi/2) J_{e0}(q, 0)} = A_1, \tag{8}$$

$$\frac{D_0 B_2^{(0)}(q) J'_{e0}(q, \mu_0)}{S_{e0}(q, \pi/2) J_{e0}(q, 0)} = 2A_2, \tag{9}$$

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$$\frac{D_0 B_4^{(0)}(q) J'_{e0}(q, \mu_0)}{S_{e0}(q, \pi/2) J_{e0}(q, 0)} = 4A_4, \tag{10}$$

where the prime means the first derivative with respect to μ. Following Strauss,[4] matching at a large distance from the plasma ($\mu \gg \mu_0$) and also sufficiently far from the conductors, the first term in the expansion of ψ_w with the corresponding term of ψ_v implies

$$A_2 = (I_1 + I_2)\alpha^2 e^{2\mu_0}/4cd^2 , \tag{11}$$

and substituting in Eq. (9), we obtain

$$\frac{8(I_1 + I_2)}{cd^2 k^2 D_0} = \frac{B_2^{(0)}(q) J'_{e0}(\psi, \mu_0) e^{-2\mu_0}}{q S_{e0}(q, \pi/2) J_{e0}(q, 0)} . \tag{12}$$

We have numerically evaluated expression (12) as a function of q, and plotted it versus the elongation of the equilibrium (which is a function of q) (see Fig. 2).

As can be seen from Fig. 2, assuming $(I_1 + I_2)$, d and k^2 fixed, the transcendental equation (12) admits solutions only for values of D_0 greater than a certain $D_{\min}$. For $D_0 > D_{\min}$, there are two possible values of q which correspond to different values of μ_0 and different elongations. The two equilibria correspond to the same values of the excluded poloidal flux and the semi-axes are well determined if k^2 is specified. For $D_0 = D_{\min}$ there is only one solution of Eq. (12), corresponding to $q = 4.88$, $\mu_0 = 0.39$, and $b/a = 2.67$, which can be interpreted as a bifurcation point.

If one considers the next term in the Taylor expansion of ψ_w and matches it with the corresponding term of ψ_v, the following relation must be satisfied

$$A_4 = (\alpha^4 e^{-4\mu_0}/128cd^4)(I_1 - I_2). \tag{13}$$

Using Eqs. (9), (10), (11) and the relation $\alpha^2 e^{2\mu_0}/4 = (a + b)^2$, it is possible to obtain

$$(I_1 - I_2)\, \frac{(a + b)^2}{4d^2} = \frac{B_4^{(0)}(q)}{B_2^{(0)}(q)}\, (I_1 + I_2). \tag{14}$$

$B_4^{(0)}/B_2^{(0)}$ is always negative and is an increasing function of q (for example, see Ref. 8). At the bifurcation point $B_4^{(0)}/B_2^{(0)} \sim -2/9$ and we see that at the plasma boundary the $\cos 4\theta$-term in the ψ_w expansion must be about $-1/9$ of the $\cos 2\theta$-term. Since we keep $I_1 + I_2$ positive, this

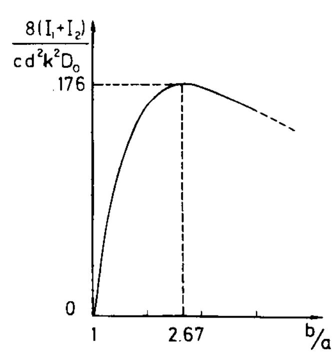

FIG. 2. Plot of $8(I_1 + I_2)/cd^2k^2D$ versus b/a for a diffuse current profile.

implies that to elongate the cross-section I_2 must be increased and I_1 reduced (and for large elongation, reversed). For instance, at the bifurcation point, for a realistic value of $(a + b)^2/d^2 = 8/45$, it follows that $I_1 \sim -2I_2/3$.

In previous studies on the subject,[4-6] I_1 has been considered to be equal to I_2. With the present model, this implies that the boundary contour cannot be an ellipse, and ψ_{p1} must be a superposition of various Mathieu harmonics. In this case we assume that ψ_{p1} is given by

$$\psi_{p1} = \frac{D_0 S_{e0}(q, \theta) J_{e0}(q, \mu)}{S_{e0}(q, \pi/2) J_{e0}(q, 0)} + \delta\psi_{p1}(\theta, \mu), \tag{15}$$

where $\delta\psi_{p1}(\theta, \mu)$ is a perturbation solution which becomes important near μ_0, where $J_{e0} = 0$. Correspondingly, ψ_v must be of the general form given in (5), with $E_4 = -A_4$. If we assume that the boundary contour is of the type $\mu = \mu_0 + \delta\mu \cos 2\theta$, we can express the boundary conditions as

$$\frac{\partial \psi_{p1}}{\partial \mu}\Big|_{\mu=\mu_0} \delta\mu \cos 2\theta + \psi_{p1}\big|_{\mu=\mu_0}$$

$$= \frac{\partial \psi_v}{\partial \mu}\Big|_{\mu=\mu_0} \delta\mu \cos 2\theta + \psi_v\big|_{\mu=\mu_0} = 0. \tag{16}$$

Up to terms in $\cos 4\theta$ for ψ_v, condition (16) gives

$$A_0 + A_1\mu_0 + \delta\mu A_2 = 0,$$

$$A_1 \delta\mu + 2A_4 + E_2 = 0,$$

$$A_2 \delta\mu - A_4 = 0.$$

From the last equation we obtain $\delta\mu = A_4/A_2$; an estimate of A_4/A_2 can be obtained by equating the normal derivatives of ψ_{p1} and ψ_v at $\mu = \mu_0$. At zero order in the perturbation, this gives

$$\delta\mu \sim B_4^{(0)}(q)/2B_2^{(0)}(q),$$

from which it results that $\delta\mu$ is negative and increases with q (Ref. 8). At the bifurcation point $\delta\mu \sim -0.1$, and correspondingly, $b/a = \cosh(\mu_0 - \delta\mu)/\sinh(\mu_0 + \delta\mu) \sim 2$. Thus, the absence of the $\cos 4\theta$ term reduces the elongation of the configuration.

It has been shown that, by a suitable choice of the current in the external conductors, a bifurcation of the equilibrium solutions for an elliptical plasma column with a diffuse current profile in an external quadrupole magnetic field exists, and occurs at $b/a = 2.67$, which is lower than the value corresponding to the flat current model.[4] The requirement of reversing I_1 and increasing I_2 in elongating the plasma cross section avoids the contact of the separatrix with the plasma, as was the case in the numerical work of Thomas and Haas[6] for $I_1 = I_2$, and allows the attainment of elongations greater than $b/a = 2.1$.

It has also been shown that for I_1 identical to I_2, the boundary contour cannot be an ellipse, but should resemble a rhomboidal ellipse. In that case, the ratio of the semi-axes corresponding to the bifurcation should be sharply reduced to a value of about 2. It is interesting to note that for $I_1 = I_2$, and for a δ-function distribution of the axial current density, there is no bifurcation, as shown by Papaloizou *et al.*[5]: Since this distribution corresponds to the particular case for n tending to infinity, of the general assumption $j_z \propto \psi_{p1}^n$ and taking into account the fact that for $n = 0$ there is bifurcation of the equilibrium, they inferred that there should be a critical value of n above which bifurcation cannot occur. From our results, it can be concluded that this critical value should be greater than one.

An interesting observation that arises from the present work is that in order to obtain large elongation, it is necessary to push the plasma from all sides with different degrees of compression.

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